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FAMILIES OF LIFE DISTRIBUTIONS CHARACTERIZED BY TWO MOMENTS

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FAMILIES OF LIFE DISTRIBUTIONS CHARACTERIZED BY TWO MOMENTS

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ABSTRACT

We consider several classical notions of partial orderings among life distributions which have been used to describe ageing properties, tail domination, etc. We show that if a distribution G dominates another distribution F in one of these partial orderings, and if two moments of G agree with those of F, including the moment that describes the partial ordering, then $G = F$. This leads to a characterization of the exponential distribution among HNBUE and HNWUE life distribution classes, and thus extends the results of Basu and Bhattacharjee (1984) and rectifies an error in that paper.

1. Introduction. We consider only life distributions in this paper, i.e. only non-negative random variables and their distributions. Throughout this paper X, Y, Λ are non-negative random variables with distribution functions F, G and H, respectively, and $H(x) = 1 - \exp(-x)$. Thus Λ is the exponential random variable with parameter 1. The survival function of X will be denoted by $\bar{F}$ where $\bar{F}(x) = 1 - F(x)$. Similarly, $\bar{G}$ will denote the survival function of Y. We begin with the definitions of several classical partial orderings among random variables.

Definition 1.1. We say that $X \geq^{st} Y$ if

$$\int_t^\infty dF(x) \geq \int_t^\infty dG(x) \text{ for each } t \geq 0. \tag{1.1}$$

Definition 1.2. Let $p > 0$. We say that $X \geq^p Y$ if

$$E(X^p) = E(Y^p), and \tag{1.2}$$

$$\int_t^\infty x^{p-1}\bar{F}(x)dx \geq \int_t^\infty x^{p-1}\bar{G}(x)dx \quad \text{for each } t \geq 0. \tag{1.3}$$

As is customary we will also transfer the notion of partial orderings to distribution functions and say that $F \geq^{st} G$ if $X \geq^{st} Y$ and $F \geq^p G$ if $X \geq^p Y$, etc.

The partial ordering $\geq^p$ is related to the partial ordering $\geq^{st}$ as shown below. Define random variables W and Z associated with X and Y as follows:

$$P(W \geq t) = p\int_t^\infty x^{p-1}\bar{F}(x)dx \Big/ E(X^p) \tag{1.4}$$

and

$$P(Z \geq t) = p\int_t^\infty x^{p-1}\bar{G}(x)dx \Big/ E(Y^p). \tag{1.5}$$

Notice that $X \geq^p Y$ if and only if $E(X^p) = E(Y^p)$ and $W \geq^{st} Z$.

Let $\succeq$ be a partial ordering and let K be a distribution function. The upper class $U(K, \succeq)$ and the lower class $L(K \succeq)$ based on K are defined by $U(K, \succeq) = \{L : \quad L \succeq K\}$ and $L(K, \succeq) = \{L : \quad K \succeq L\}$

Many of the ageing classes appearing in the literature in Reliability Theory are upper or lower classes determined by the exponential distribution and an appropriate partial ordering. Thus the SU, IFR, DMRL, IFRA, NBU, NBUE and HNBUE classes of distributions are lower classes determined by the exponential distribution and a chain of progressively weaker partial orderings. (A similar statement can be made for the duals of the above classes.) For instance, consider the HNBUE class defined in Rolski (1975) and Klefsjö (1982). Let the random variable X with distribution function F satisfy $E(X) = 1$. Then X is HNBUE if and only if $\frac{1}{t} \log \int_t^\infty \bar{F}(x)dx \geq 1$, or if and only if $\int_t^\infty \bar{F}(x)dx \leq e^{-t} = \int_t^\infty e^{-x}dx$, i.e. if and only if $F \leq^1 H$. Thus the HNBUE class is precisely the lower class $L(H, \geq^1)$. Similarly, the dual class HNWUE can be seen to be the upper class $U(H, \geq^1)$.

2. Results. We lead to the main theorem, Theorem 2.4, of this paper in a transparent way through easily proved results.

Theorem 2.1. Let $X \geq^{st} Y$. If $E(X)$ is finite and $E(X) = E(Y)$ then $X =^{st} Y$.

Proof: There are many ways to prove this well-known theorem. The conceptually simple way is to view X and Y as random variables on a single probability space with $X(\omega) \geq Y(\omega)$ for almost all ω, and appealing to a standard result in measure theory. A direct proof notes that $\bar{F}(x) \geq \bar{G}(x)$ for all x and $0 = E(X) - E(Y) = \int_0^\infty (\bar{F}(x) - \bar{G}(x))dx$. This implies that $\bar{F}(x) = \bar{G}(x)$ for all x and $X =^{st} Y$. □

A minor extension of Theorem 2.1 is the following.

Theorem 2.2. Let $X \geq^{st} Y$. For some $\alpha \neq 0$, let $E(X^\alpha)$ be finite and $E(X^\alpha) = E(Y^\alpha)$. Then $X =^{st} Y$.

Proof: Let $\alpha > 0$. Let $W = X^\alpha$, $Z = Y^\alpha$. Then $W \geq^{st} Z$, $E(W)$ is finite and $E(W) = E(Z)$. Thus $W =^{st} Z$ and $X =^{st} Y$. If $\alpha < 0$, define $W = Y^\alpha$ and $Z = Y^\alpha$ and proceed as above to obtain the same conclusion. □

Our main results are the next two theorems which extend the idea used in Theorem 2.2.

Theorem 2.3. Let $p > 0$ and $X \geq^p Y$. Suppose that for some $r \neq 0,\ r \neq p$, we have that $E(X^r)$ is finite and $E(X^r) = E(Y^r)$. Then $X =^{st} Y$.

Proof: Note that $X \geq^p Y$ implies $E(X^p) = E(Y^p)$. Suppose that $r > p$. Define random variables W and Z by the survival functions in (1.4) and (1.5). Then $W \geq^{st} Z$. Furthermore, since $r \neq 0$,

$$E(W^{r-p}) = \frac{E(X^r)}{rE(X^p)} = \frac{E(Y^r)}{rE(Y^p)} = E(Z^{r-p}).$$

From Theorem 2.2 it now follows that $W =^{st} Z$ and

$$p\int_t^\infty x^{p-1}\bar{F}(x)dx = p\int_t^\infty x^{p-1}\bar{G}(x)dx \text{ for all } t > 0.$$

This implies that $\bar{F}(x) = \bar{G}(x)$ for all x and $X =^{st} Y$.

If $p > r$, we interchange $\bar{F}$ and $\bar{G}$ in definitions (1.4) and (1.5) of W and Z and proceed as above to obtain the same conclusion. □

Theorem 2.3 leads to our other main result which is given below.

Theorem 2.4. Let X be a non-degenerate random variable with distribution function F belonging to $\mathcal{A}$ where $\mathcal{A}$ is one of the standard ageing class in Reliability, that is, $\mathcal{A}$=SU, IFR, DMRL, IFRA, NBU, NBUE or HNBUE. Then F is exponential if and only if

(2.1) $$E(X^r) = \Gamma(r+1)(E(X))^r \text{ for some } r \text{ in } (-1,\infty) \text{ with } r \neq 0 \text{ and } r \neq 1.$$

Proof: Without loss of generality, we can assume that X is HNBUE, the largest of the ageing classes above and $EX = 1$, since the HNBUE property is scale-invariant. Thus F belongs to $L(H, \geq^1)$ and $\Lambda \geq^1 X$ where H is the distribution function of an exponential r.v. Λ with parameter 1. Condition (2.1) now says that $E\Lambda^r = EX^r$ with $E\Lambda^r < \infty$ for some r satisfying $-1 < r < \infty,\ r \neq 0,\ r \neq 1$. From (2.3), it follows that $X =^{st} \Lambda$. □

Remark 1: Basu and Bhattacharjee (1984) in a paper on the weak convergence within the HNBUE class, first stated and proved Theorem 2.4 with $\mathcal{A}$ =HNBUE, $0 < r < \infty, r \neq 1$. However, their argument for the case $0 < r < 1$ required that the function $\psi(t) = \int_0^t (\bar{F}(x) - e^{-x})dx$ be a distribution function on $(0, \infty)$. The function $\psi(t)$ need not be a distribution function since the HNBUE property of F (with unit mean) implies only that $\psi(t) \geq 0$. Our proof, which draws on the more general result in Theorem 2.3, remedies this gap in their proof.

Also, since $E\Lambda^r = \Gamma(r+1) < \infty$ for $r > -1$ the admissible range of r for their theorem is extended to $(-1, \infty)\backslash(\{0\} \bigcup \{1\})$. Note also that the $EX^r < \infty$ for any $r > 0$ with any of the ageing classes under consideration.

Remark 2: It is clear that Theorem 2.4 remains true when $\mathcal{A}$ is any of the anti-ageing classes with finite means which are dual to the ageing classes stated in Theorem 2.4. This is so because the upper class $U(H, \geq^1)$ define the largest of these anti-ageing classes (HNWUE) with mean one.

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